Complex Numbers

Imaginary Number: $i=\sqrt{-1}$ and $i^2=-1$

Complex Numbers: The set of all numbers of the form $a+bi$ where a and b are real numbers.

a is called the **real part** and b is called the **imaginary part**. If $b\neq 0$, then $a+bi$ is an **imaginary number.** The form $a+bi$ is called the **standard form** of a complex number.

Examples: Determine whether each complex number is real or imaginary and write it in standard form.

a) $4i$ b) $3-6i$ c) 5 d) $\dfrac{i-3\pi}{4}$

Addition, Subtraction, and Multiplication of Complex Numbers.

$(a+bi)+(c+di)=(a+c)+(b+d)i$

$(a+bi)-(c+di)=(a-c)+(b-d)i$

$(a+bi)(c+di)=(ac-bd)+(bc+ad)i$ or use FOIL.

Examples: Perform the indicated operations.

a) $(6+2i)+(4-3i)=$ b) $(7-4i)-(-2+8i)=$

c) $(6+5i)(8+3i)=$ d) $(1-i)(4+i)=$

Powers of *i*:

Since $i^2=-1$, $i^3=i^2\cdot i=-1\cdot i=-i$, and $i^4=i^2\cdot i^2=-1\cdot -1=1$

The first eight powers are listed here:

$i^1=i$ $i^5=i$

$i^2=-1$ $i^6=-1$

$i^3=-i$ $i^7=-i$

$i^4=1$ $i^8=1$

The powers of i continue in this pattern.

Examples: Simplify the power of i.

a) $i^{35} =$ b) $i^{29} =$ c) $i^{98} =$ d) $i^{48} =$

Theorem: If a and b are real numbers, then the product of $a+bi$ and its conjugate $a-bi$ is the real number a^2+b^2. $(a+bi)(a-bi)=(a^2+b^2)$

Examples: Find the product of the complex number and its conjugate.

a) $3-7i$ b) $2+9i$ c) i

Examples: Write each quotient in the form $a+bi$.

a) $\frac{6-2i}{3} =$ b) $\frac{2}{8+9i} =$ c) $\frac{4-5i}{3+2i} =$

Roots of Negative Numbers

For any positive real number b, $\sqrt{-b} = i\sqrt{b}$.

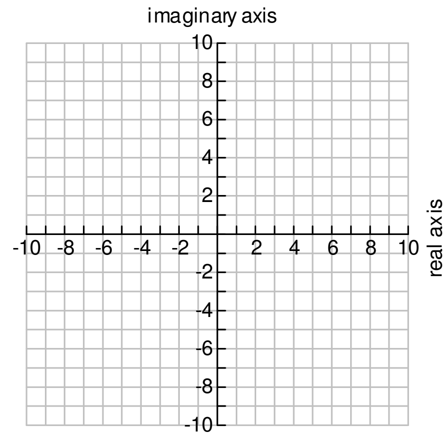
Examples: Write each expression in the form $a+bi$, where a and b are real numbers.

a) $\sqrt{-5}+\sqrt{-8} =$ b) $\sqrt{-20}\left(\sqrt{-6}-\sqrt{-4}\right) =$

c) $\frac{-2+\sqrt{-48}}{2} =$

Example: Does the complex number $x = 1+3i\sqrt{2}$ satisfies the equation $x^2-2x+4=0$?

Trigonometric Form of Complex Numbers



The complex number $a+bi$ can be thought of as an ordered pair (a,b). We graph it on the **complex plane** where the horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis.**

Absolute Value or Modulus: $|a+bi| = \sqrt{a^2+b^2}$. (The distance between the number and the origin on the complex plane.)

Examples: Graph each complex number and find its absolute value.

a) $5-i$

b) $-6+2i$

Trigonometric Form of a Complex Number

If $z = a+bi$ is a complex number, then the trigonometric form of z is

$$z = r(\cos\theta + i\sin\theta), \text{ sometimes abbreviated } z = r\operatorname{cis}\theta,$$

where r is called the **modulus** and θ is called the **argument,** defined as the angle in standard position whose terminal side contains the point (a,b).

$$r = \sqrt{a^2+b^2}$$
$$a = r\cos\theta \text{ and } b = r\sin\theta.$$

We usually use the smallest possible nonnegative angle for θ.

Examples: Write each complex number in trigonometric form. Express θ in degrees.

a) $-2\sqrt{3}+2i$

b) $5-4i$

Example: Write the complex number $12\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ in the form $a+bi$.

Product and Quotient of Complex Numbers

If $z_1=r_1\left(\cos\theta_1+i\sin\theta_1\right)$, and $z_2=r_2\left(\cos\theta_2+i\sin\theta_2\right)$, then

$$z_1z_2=r_1r_2\left[\cos\left(\theta_1+\theta_2\right)+i\sin\left(\theta_1+\theta_2\right)\right]$$

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}\left[\cos\left(\theta_1-\theta_2\right)+i\sin\left(\theta_1-\theta_2\right)\right]$$

Examples: Find the product and quotient using trigonometric form.

$$z_1=4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right),\quad z_2=8\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

a) Find z_1z_2

b) Find $\frac{z_1}{z_2}$

Complex Conjugates

The conjugate of $r\left(\cos(\theta)+i\sin(\theta)\right)$ is $r\left(\cos(-\theta)+i\sin(-\theta)\right)$

A complex number times its conjugate equals r^2.

Proof: $r\left(\cos\theta+i\sin\theta\right)\cdot r\left(\cos(-\theta)+i\sin(-\theta)\right)$

$=r^2\left(\cos(\theta-\theta)+i\sin(\theta-\theta)\right)$

$=r^2\left(\cos 0+i\sin 0\right)$

$=r^2\left(1+0i\right)=r^2$

Example: Find the product of the following and its conjugate: $6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

Powers and Roots of Complex Numbers

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is any positive integer, then

$$z^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

Examples:

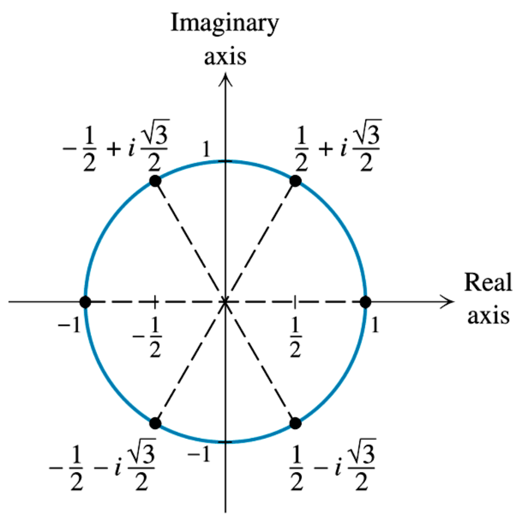
a) Simplify $(1+i)^6$.

b) Simplify $\left(\sqrt{3}-i\right)^4$.

Roots of a Complex Number

How many square roots does 4 have?
How many square roots does -9 have?
How many sixth roots does 1 have?

It turns out that 1 has 6 sixth roots, and they are distributed evenly around the complex plane.



Sixth Roots of Unity

The complex number $a+bi$ is an nth root of the complex number z if $\left(a+bi\right)^n = z$.

For any positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots given by the expression $r^{1/n}\left[\cos\left(\frac{\theta+360°k}{n}\right)+i\sin\left(\frac{\theta+360°k}{n}\right)\right]$ for $k = 0,1,2,\ldots,n-1$.

In radians, the roots are given by $r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$ for $k = 0,1,2,\ldots,n-1$.

The first of the n roots has an argument of $\frac{\theta}{n}$, and the other roots are spaced $\left(\frac{360}{n}\right)^\circ$ apart.

(The circle is divided evenly into n pieces.)

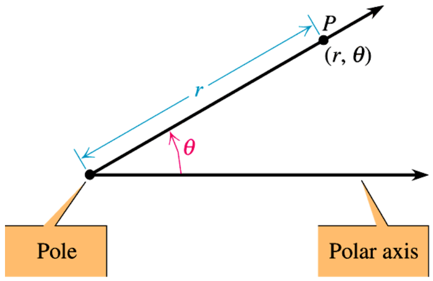
Examples:

a) Find all of the fourth roots of the complex number $-8-8i\sqrt{3}$.

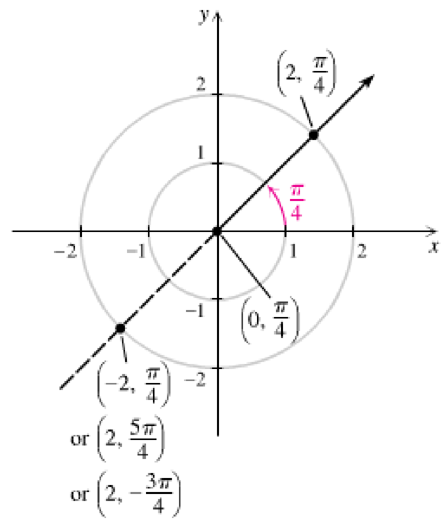
b) Find all the cube roots of 125.

c) Find all complex solutions to $x^4 - 81 = 0$.

Polar Equations



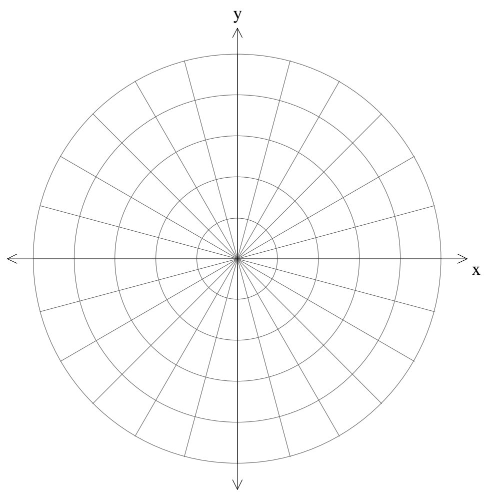
The polar coordinate system is based on a fixed point called the **pole** and a fixed axis called the **polar axis.** Points are represented by ordered pairs in the form (r,θ), where r is the **directed distance** from the pole and θ is an angle whose initial side is the polar axis and whose terminal side contains the point. Typically, we choose the origin as the pole and the positive x-axis as the polar axis.



*To graph $(-r,\theta)$, you move in the opposite direction you would move to graph (r,θ).

Polar coordinates are not unique. The points $\left(-2,\frac{\pi}{4}\right)$, $\left(2,\frac{5\pi}{4}\right)$, and $\left(2,-\frac{3\pi}{4}\right)$ all name the same point.

Examples: Plot the points whose polar coordinates are given.
$A\left(3,\frac{\pi}{3}\right)$, $B\left(-1,\frac{\pi}{6}\right)$, $C\left(2,-\frac{7\pi}{4}\right)$, $D\left(-5,-\frac{3\pi}{4}\right)$, $E\left(4,\frac{\pi}{2}\right)$, $F\left(-3,\frac{2\pi}{3}\right)$



Polar-Rectangular Conversion Rules

- To convert (r,θ) to rectangular coordinates (x, y), use $x = r\cos\theta$ and $y = r\sin\theta$.

- To convert (x, y) to polar coordinates (r,θ), use $r = \sqrt{x^2 + y^2}$ and any angle θ in standard position whose terminal side contains (x, y).

Examples:

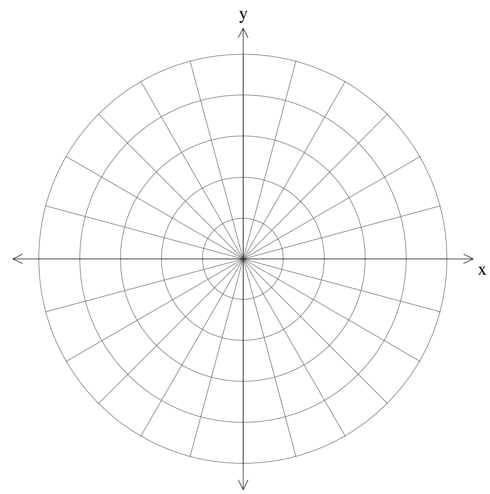
a) Convert $(3, 45°)$ to rectangular coordinates.

b) Convert $\left(-2, 2\sqrt{3}\right)$ to polar coordinates.

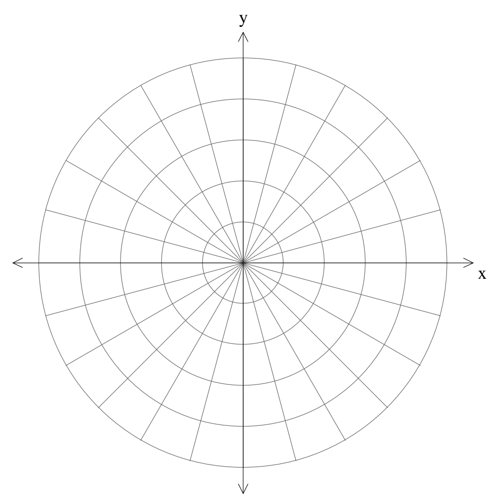
Graphing Polar Equations

Examples: Sketch the graphs of the following:

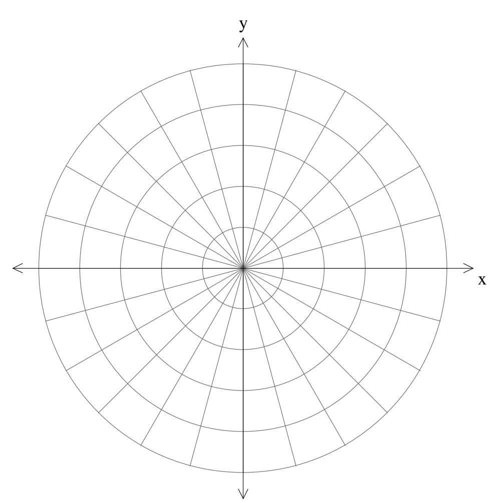
a) $r = 4\sin\theta$



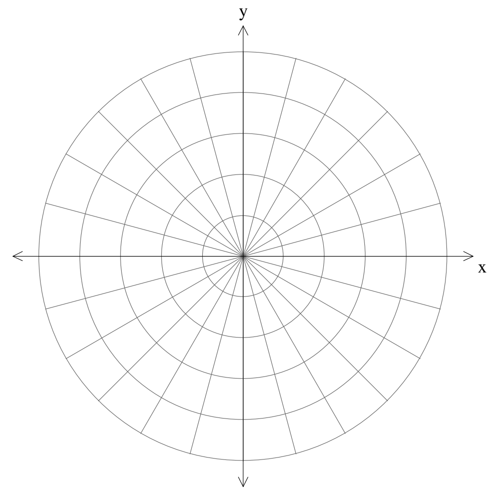
b) $r = 3\cos(2\theta)$



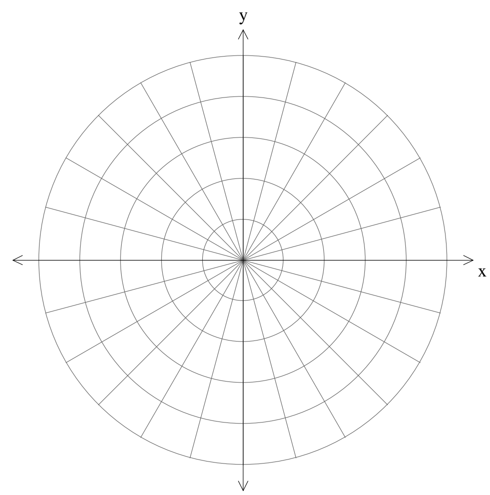
c) $r = 2 + 3\sin\theta$



d) $r^2 = 4\sin(2\theta)$



e) $r = 5\sin(4\theta)$



Examples: Convert equations from polar to rectangular form.

a) Convert $r = 3\sin\theta$ to a rectangular equation.

b) Convert $r = \dfrac{4}{1+\sin\theta}$ to a rectangular equation.

c) Convert $r = 5\sec\theta$ to a rectangular equation.

d) Convert $r = \frac{\pi}{3}$ to a rectangular equation.

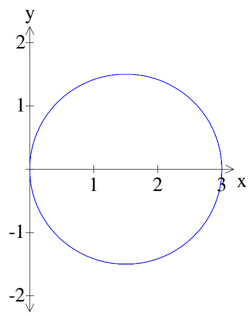
Examples: Convert equations from rectangular to polar form.
a) Convert $y = 7$ to a polar equation.

b) Convert $y = -2x + 5$ to a polar equation.

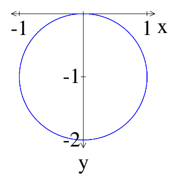
c) Convert $x^2 + (y-1)^2 = 1$ to a polar equation.

1) Lines through the origin are of the form $\theta = \alpha$.
Vertical lines are of the form $r = a\sec\theta$.
Horizontal lines are of the form $r = a\csc\theta$.

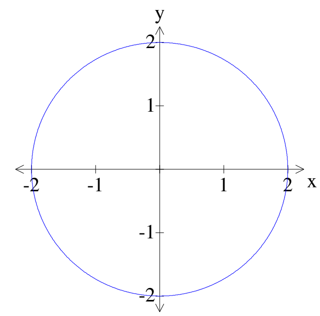
2) Circles come in three forms: $r = a\cos\theta$, $r = a\sin\theta$, and $r = a$.



$r = 3\cos\theta$

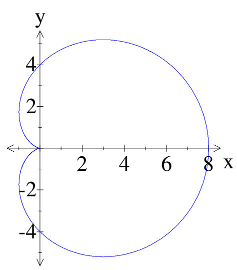


$r = -2\sin\theta$

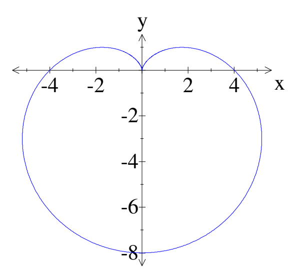


$r = 2$

3) Cardioids have the form $r = a \pm a\cos\theta$ or $r = a \pm a\sin\theta$.
Cardioids pass through the pole.

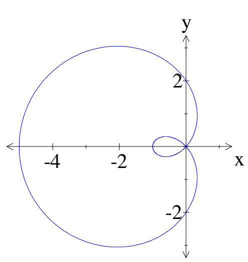


$r = 4 + 4\cos\theta$

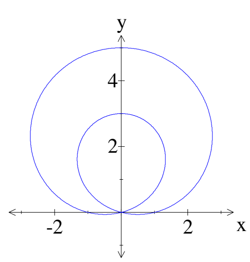


$r = 4 - 4\sin\theta$

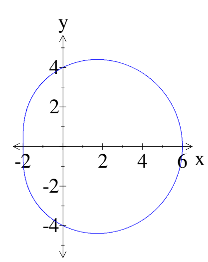
4) Limaçons have the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$.
Limaçons have an inner loop if $0 < a < b$ and have no inner loop if $0 < b < a$.
The graph of a limaçon with an inner loop passes through the pole twice.
The graph of a limaçon with no inner loop does not pass through the pole.



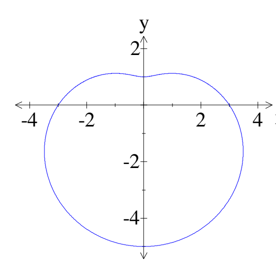
$r = 2 - 3\cos\theta$



$r = 1 + 4\sin\theta$

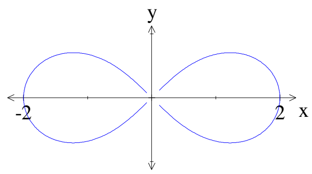


$r = 4 + 2\cos\theta$

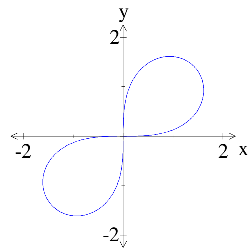


$r = 3 - 2\sin\theta$

5) Lemniscates have the form $r^2 = a^2 \cos(2\theta)$ or $r^2 = a^2 \sin(2\theta)$.

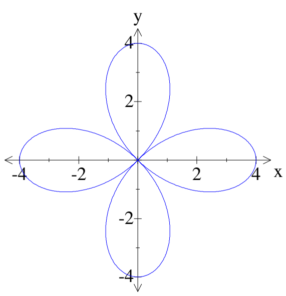


$r^2 = 4\cos(2\theta)$

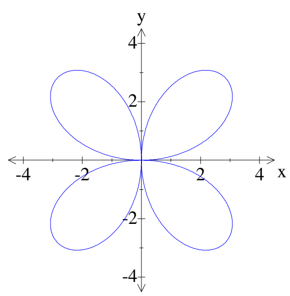


$r^2 = 4\sin(2\theta)$

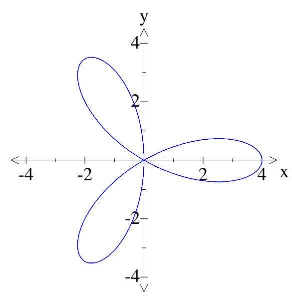
6) Roses have the form $r = a\cos(n\theta) + b$ and $r = a\sin(n\theta) + b$.
If n is even, there are $2n$ loops in the rose.
If n is odd, there are n loops in the rose.



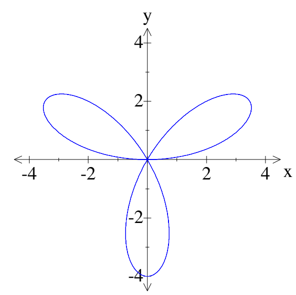
$r = 4\cos(2\theta)$



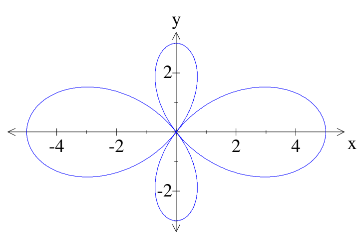
$r = 4\sin(2\theta)$



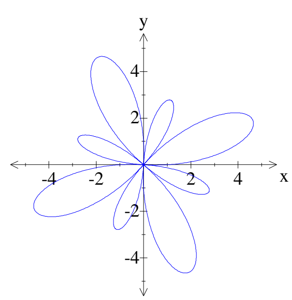
$r = 4\cos(3\theta)$



$r = 4\sin(3\theta)$



$r = 4\cos(2\theta) + 1$



$r = 4\sin(4\theta) + 1$

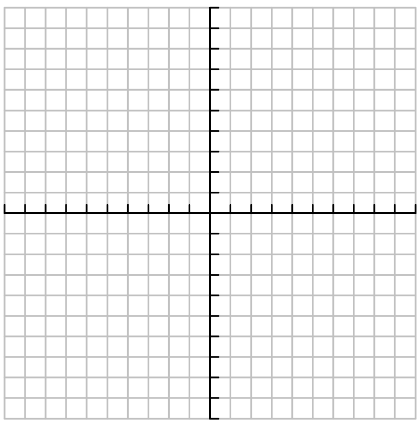
Parametric Equations

Sometimes, it is convenient to express both x and y as functions of a third variable, t. If $f(t)$ and $g(t)$ are both functions of t, where t is some interval of real numbers, then the equations $x = f(t)$ and $y = g(t)$ are called **parametric equations.** The variable t is called the **parameter.** If we think of t as time, then we know when each point of the graph is plotted.

Graphing Parametric Equations

1. Make a t, x, y table for the two equations.
2. Plot the ordered pairs of values of x and y.
3. Mark the **orientation** of the curve by using arrows to show the direction of the graph.

Example: Graph the parametric equations $x = t + 5$ and $y = 2t - 1$ for t in $[0,5]$.



Eliminating the Parameter

1. Set one equation equal to t.
2. Substitute that equation in for t in the other equation.
3. Sometimes it is more convenient to use a trigonometric identity to eliminate the parameter.

Examples: Eliminate the parameter and identify the graph of the parametric equation.

a) $x = 4t - 9,\ y = -t + 1,\ -\infty < t < \infty$

b) $x = 2\sqrt{t},\ y = 8t + 6,\ \ 0 \le t < \infty$

c) $x = 5\sin t,\ y = 5\cos t,\ -\infty < t < \infty$

d) $x = 2\sin\theta,\ y = 3\cos\theta,\ \ -\infty < \theta < \infty$

Writing Parametric Equations for Line Segments

1. Write both parametric equations as linear functions: $x = m_1 t + b_1$, and $y = m_2 t + b_2$.
2. Substitute x and t values into the x equation to create a system of equations you can solve for m_1 and b_1.
3. Substitute y and t values into the y equation to create a system of equations you can solve for m_2 and b_2.

Examples:

Write parametric equations for the line segment starting at $(1,2)$ with $t = 0$ and ending at $(8,10)$ with $t = 1$.

Write parametric equations for the line segment starting at $(-2,4)$ with $t = 3$ and ending at $(5,-9)$ with $t = 7$.

Writing Parametric Equations for a Polar Equation

Use the equations $x = r\cos\theta$ and $y = r\sin\theta$. Replace r to obtain the parametric equations. When converting polar equations to parametric equations, θ acts as the parameter.

Example: Write parametric equations for the polar equation $r = 3\cos\theta$.